17.7. SL problem
17.8. Periodic & singular SL problem
17.9. FI

Quiz #2 4/28 (Tu)

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17.7 Sturm-Liouville problem

I. Definition

$$[p(x)y]' + q(x)y + \lambda\,\omega(x)y = 0, \text{ on } [a,b]$$

with BCs.

$$\begin{cases} \alpha y(a) + \beta y'(a) = 0 \\ \gamma y(b) + \delta y'(b) = 0 \end{cases}$$

→ α, β are NOT zero simultaneously

γ, δ ~

p, p', q, ω are continuous on $[a,b]$

and $p(x)$, $w(x) > 0$.

weight fxn.

For any λ provides the corresponding solution ϕ, we call λ eigenvalue, $\phi(x)$ eigenfxn!! "characteristic"

II why SL problems?

☆. PDEs $\xrightarrow{SOV}$ 2 ODEs.

$\downarrow$

SL problem.

2. $\phi_n(x) \rightarrow$ OG basis $\rightarrow f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$

$$a_n = \frac{\langle f(x), \phi_n(x) \rangle}{\langle \phi_n(x), \phi_n(x) \rangle}.$$

<Ex.>

Consider the case, $y'' + \lambda y = 0$ —(*)

$(0 < x < L)$; $y(0) = y(L) = 0$. Find y

Sol.

$p(x) = 1$, $q(x) = 0$, $w(x) = 1$.

(*) is an SL problem!

7.7. SL problem

7.8. Periodic & singular SL problem

7.9. FI

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set $y = e^{rx} \to r^2 + \lambda = 0 \to r = \pm\sqrt{\lambda}\, i$

$$\therefore y(x) = \begin{cases} A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x, & \lambda \neq 0 \\ C + Dx, & \lambda = 0 \end{cases} \qquad \text{— (1(a,b))}$$

For 1(b),

$y(0) = 0 = C$; $y(L) = 0 = DL$ $\therefore D = 0$

$\to$ leads to $y(x) = 0$ "trivial sol."

$\therefore \lambda = 0$ is NOT an eigenvalue!

For 1(a), homogeneous!

$y(0) = 0 \quad \therefore 0 = A \cdot 1 + B \cdot 0 \to A = 0$

$y(L) = 0 \quad \therefore 0 = A\cos\sqrt{\lambda}L + B\sin\sqrt{\lambda}L$

$$\begin{vmatrix} 1 & 0 \\ \cos\sqrt{\lambda}L & \sin\sqrt{\lambda}L \end{vmatrix} = 0 \quad \therefore \sin\sqrt{\lambda}L = 0$$

assure non-trivial sol.

$$\sqrt{\lambda}\,\mathcal{L} = \cancel{0},\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi,\ \cdots,\ \pm n\pi.$$

$$\lambda_n = \left(\frac{n\pi}{\mathcal{L}}\right)^2 \longrightarrow \text{eigenvalue}$$

$$\phi_n(x) = \sin\sqrt{\lambda_n}\,x = \underbrace{\sin\frac{n\pi x}{\mathcal{L}}}_{\Downarrow\ \text{Eigen fxns!}} \qquad (n = 1, 2, 3, \cdots)$$

III. SL Theorem & Non-negative λ's.

1. SL Theorem.

(i) All λ's are <u>REAL</u>!!

(ii) λ's are simple $(\lambda \longleftrightarrow \phi)$

(iii) If $\lambda_j \neq \lambda_k$, then $\underline{\langle \phi_j, \phi_k \rangle} = 0$

$$\int_a^b \phi_j \phi_k\, w(x)\, dx$$

9.7. SL problem

9.8. Periodic & singular SL problem

9.9. FI

Quiz #2 4/28 (Tu)

TA Hrs 4/21,

(iv) Eigenfxn Expansion

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

$$a_n = \frac{\langle f(x), \phi_n(x) \rangle}{\langle \phi_n(x), \phi_n(x) \rangle}$$

III.

2. Non-negative λ's.

if $q(x) \le 0$ on $[a,b]$, and $\left[p(x)\, \phi_n(x)\, \phi_n'(x) \right] \Big|_a^b \le$

then not only is λ_n real, $\lambda_n \ge 0$

$$\left[\phi(L)\, \phi'(L) - \phi(0)\, \phi'(0) \right] = 0 - 0 = 0 \le 0$$

<Ex>.

$$y'' - 2y' + \lambda y = 0 , \quad (0 < x < \pi)$$

$$y(0) = y(\pi) = 0$$

find $\phi_n(x) = ?$

$$[p(x)y']' + q(x)y + \lambda w(x) y = 0$$

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7. SL
problem

8. Periodic
singular
problem

. FI

...iz #2 ...
...8 (Tu)

...rs 4/21,

17.8. Periodic & Singular SL problems.

I.

1. Periodic BCs

$$y(a) = y(b) = ? \quad ; \quad y'(a) = y'(b) = ?$$

2. Singular BCs

$$p(x) - \begin{cases} x = a, & p(x) = 0 \\ x = b, & p(x) = 0 \\ x = a, b, & p(x) = 0 \end{cases}$$

II. S.L. theorem.

1. λ's are real.

2. λ's are <u>NOT</u> necessary simple!

3 orthogonality stands!

4. Eigenfxn Expansion $\rightarrow f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$

<Ex>.

A Bessel Eqn. $(xy')' + \lambda \cdot xy = 0$, $(0 < x < L)$

$y(0)$ bounded, $y(L) = 0$

<u>Sol.</u>

$$xy'' + y' + \lambda xy = 0 \rightarrow x^2 y'' + xy' + \underline{\lambda x^2 y} = 0 \quad (1)$$

<u>PS</u> $x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \rightarrow y = A J_\nu(x) + B Y_\nu(x)$

Set $x = \alpha t \rightarrow y(x) = y(x(t)) \equiv T(t)$.

$$y' = \frac{dy}{dx} = \frac{dT}{dt}\frac{dt}{dx} = \dot{T} \cdot \frac{1}{\alpha}$$

$$= \frac{\dot{T}}{\alpha}$$

$$y'' = \frac{dy'}{dx} = \frac{1}{\alpha} \cdot \frac{d\dot{T}}{dt} \cdot \frac{dt}{dx} = \frac{\ddot{T}}{\alpha^2}$$

7.~~7~~. SL problem

7.8. Periodic singular SL problem

7.9. FI

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$$(\alpha t)^2 \cdot \frac{\ddot{T}}{\alpha^2} + (\alpha t)\frac{\dot{T}}{\alpha} + \lambda \alpha^2 t^2 T = 0.$$

$$\rightarrow t^2 \ddot{T} + t\dot{T} + \underline{\lambda \alpha^2 t^2} T = 0 \quad \text{--- (2)}$$

set $\lambda \alpha^2 = 1$. $(t^2 - 0^2)$

(2) becomes, $t^2 \ddot{T} + t\dot{T} + (t^2 - 0^2) T = 0$ — (3)

① $\lambda \neq 0$

$\therefore \; T(t) = A J_0(t) + B Y_0(t)$. ———— (4).

J: Bessel's fxn. of the 1^{st} kind

Y: ~ 2^{nd} kind.

② $\lambda = \cancel{0}$. $t^2 \ddot{T} + t \dot{T} = 0 \rightarrow T(t) = C + D \ln X$ ———— (5)

$y(0)$ bounded, $y(L) = 0$. $T(0) = C + D \ln X$

$X = \alpha t = L$

$\therefore D = 0, \; T = C$.

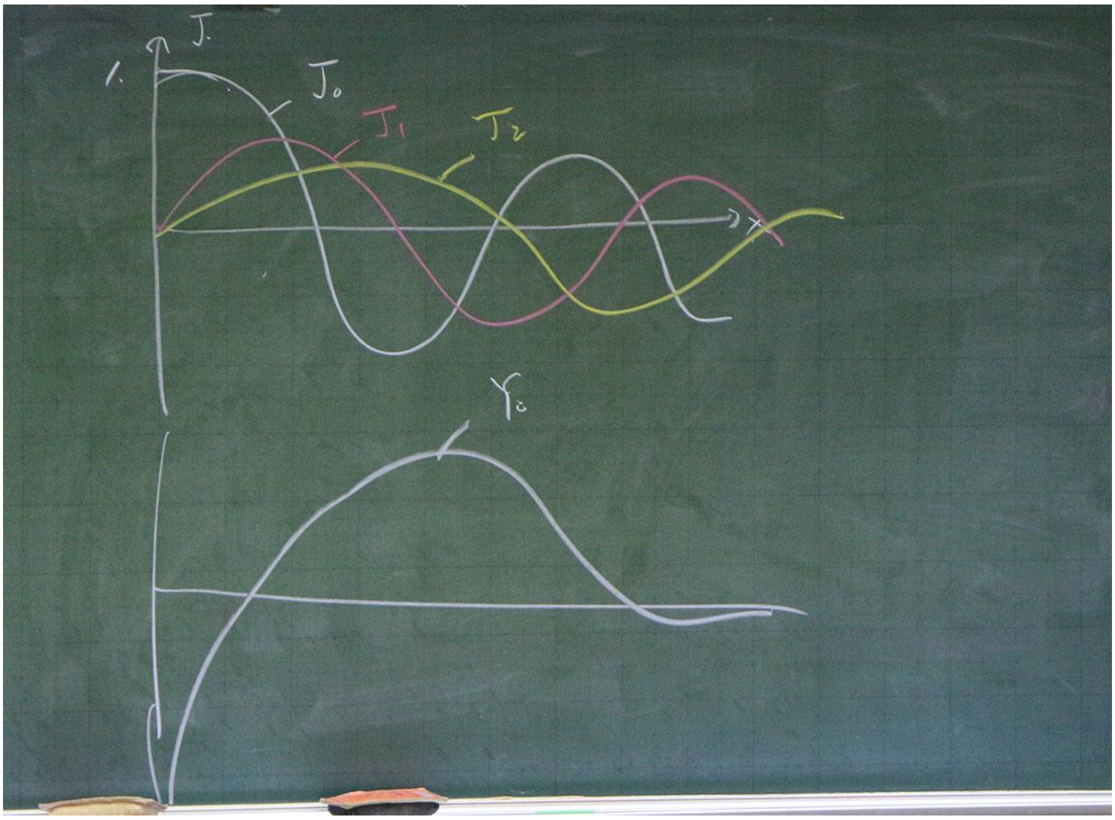
$y(L) = 0, \; T\left(\frac{L}{\alpha}\right) = 0 = C. \; \therefore C = 0$

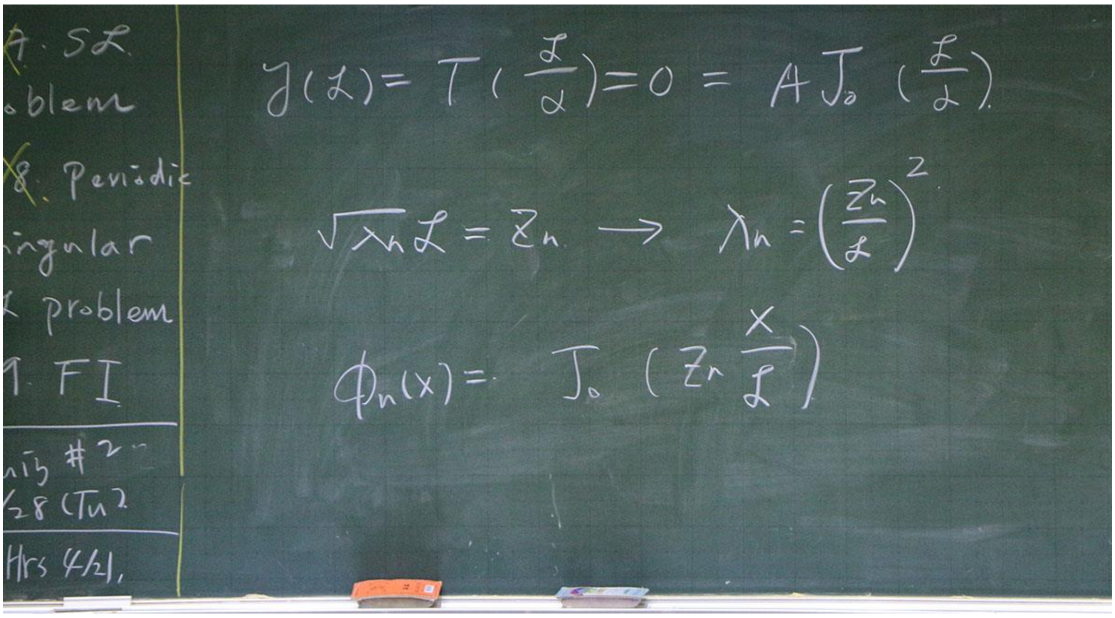
$\therefore \lambda = 0$ gives rise to a trivial sol.

For (4),

$y(0) = T(0)$ bounded, $T = A J_0(t) + B Y_0(t)$

$B = 0$


J.
1.
J_0
J_1
J_2
x
Y_0


A. SL
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8. Periodic
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9. FI
uiz #2
/28 (Tu)
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$y(\ell) = T\left(\frac{\ell}{\alpha}\right) = 0 = A J_0\left(\frac{\ell}{\alpha}\right)$
$\sqrt{\lambda_n}\,\ell = z_n \rightarrow \lambda_n = \left(\frac{z_n}{\ell}\right)^2$
$\phi_n(x) = J_0\left(z_n \frac{x}{\ell}\right)$

17-7. Fourier Integral

I. Background:

$$FS = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$\cos\frac{n\pi x}{l} \rightarrow \cos n\Delta\omega x$

$\frac{\pi}{l} = \Delta\omega$

As $l \longrightarrow \infty$, $FS\, f(x) = ?$

2. $$FI\, f(x) = \int_0^{\infty} \left[\underline{a(\omega)}\cos\omega x + \underline{b(\omega)}\sin\omega x \right] d\omega$$

where $$a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underline{f(x)\cos\omega x}\,dx \qquad \text{—— } (*)$$

$$b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx$$

Set $\frac{\pi}{l} = \underbrace{\Delta\omega}_{\text{freq.}}$ $\to \cos \boxed{\omega x}$

$$\cos\frac{n\pi x}{l} = \cos(n\Delta\omega)x$$

$$\sum_{n=1}^{\infty} f(n\Delta\omega)\,\Delta\omega \xrightarrow[\Delta\omega\to 0]{l\to\infty} \int_0^{\infty} f(\omega)\,d\omega$$

"Riemann Sum"

x	$\vec{r}$	t
ω	$\vec{K}$	ω

"frequeny"

$$e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$